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MODERATELY LARGE DEFLECTION OF ASYMMETRICALLY LAYERED ELASTIC PLATE*

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Abstract—A parametric expansion technique is used for extracting systems of two-dimensional equations governing the moderately large deflection of an elastic asymmetrically layered transversely isotropic plate from a three-dimensional partially nonlinear theory of elasticity in terms of a reference state or the Lagrangian description. The equations of the latter theory are first expressed in dimensionless form for each thin layer by means of a suitable scaling of the stresses and displacements involving the expansion parameter. Successive systems of differential equations are then obtained by equating corresponding powers of the latter after all the variables are expanded parametrically. The first such system yields a first-approximation von Kármán-type nonlinear layered plate theory when all continuity conditions at the bonding interface between layers are taken into account. For a plate made of two dissimilar layers, the corresponding differential equations in terms of the displacements defined at the middle plane of one of the layers are presented. Systematic higher-order approximations can be obtained from the next systems of equations. The first such system would exhibit the effects of transverse shear and normal stresses on the deformation.

NOTATION

γ_{ij}	symmetric strain tensor
v_i	displacement vector
s_{ij}	symmetric stress tensor measured per unit area of the undeformed body
$\tilde{s}_{*i}$	prescribed components of the stress vector per unit area of the undeformed body when referred to base vectors in the undeformed body
${}_0n_i$	unit normal to the undeformed position of a surface in the deformed body
E, E_3	Young's moduli
ν, ν_3	Poisson's ratios
G	shear modulus
H	uniform thickness of lower plating
ξ, η, ζ	dimensionless system of Cartesian coordinates for lower plating
$s_x, s_y, t,$ t_x, t_y, s	dimensionless stress components
u, v, w	dimensionless displacement components
$(\ )'$	quantities referring to the upper plating

INTRODUCTION

In recent years, as a result of an increasing volume of research devoted to the mechanics of nonconventional thin structural elements intended for various spacecraft applications, several contributions have appeared in the field of multilayered structures [1]. It appears

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that most existing theories of such composite elements have been obtained from sets of *a priori* estimates concerning the distribution of displacements and at least of some of the stresses and/or strains through the thickness, depending on the method of derivation employed, in close parallel to similar studies of conventional structures. When the governing two-dimensional equations are deduced from the leading terms of suitable parametric expansions of solutions of three-dimensional elasticity theory, however, the essence of the matter becomes the adoption of a proper scaling of the variables involved. The use of such considerations in the derivation of successive systems of differential equations governing the moderately large deflection of a homogeneous elastic plate has lately been illustrated by Ebcioglu and the author [2] through parametric expansion of the interior solution of a partially nonlinear theory of elasticity. In what follows, the same will be applied to the moderately large deflection of an asymmetrically layered transversely isotropic plate. A linear bending theory of isotropic sandwich plates based upon a similar viewpoint is due to Gerard [3].

FUNDAMENTAL EQUATIONS

The three-dimensional equations of equilibrium, in the absence of body forces, and boundary conditions corresponding to the following partially nonlinear strain–displacement relations

$$\left.\begin{aligned} \gamma_{\alpha\beta} &= \tfrac{1}{2}(v_{\alpha,\beta}+v_{\beta,\alpha}+v_{3,\alpha}v_{3,\beta}),\\ \gamma_{\alpha 3} &= \tfrac{1}{2}(v_{\alpha,3}+v_{3,\alpha}),\\ \gamma_{33} &= v_{3,3},\end{aligned}\right\} \qquad (1)$$

are [2]

$$\left.\begin{aligned} s_{\alpha\beta,\beta}+s_{\alpha 3,3} &= 0,\\ (s_{\alpha 3}+s_{\alpha\beta}v_{3,\beta})_{,\alpha}+s_{33,3} &= 0,\end{aligned}\right\} \qquad (2)$$

and

$$\left.\begin{aligned} \tilde{s}_{*\alpha} &= s_{\alpha\beta 0}n_\beta+s_{\alpha 30}n_3,\\ \tilde{s}_{*3} &= (s_{\alpha 3}+s_{\alpha\beta}v_{3,\beta})_0 n_\alpha+s_{330}n_3,\end{aligned}\right\} \qquad (3)$$

respectively. These relations are written in a Cartesian system of convected coordinates, x_i, using indicial notation and the Lagrangian description [4]. Latin indices take the values 1, 2, 3, and Greek indices take the values 1, 2; repeated indices denote summation over their respective ranges. A comma means partial differentiation with respect to the indicated coordinate.

The use of general nonlinear strain–displacement relations in the formulation of a theory of elastic plates in terms of a reference state or the Lagrangian description have previously been illustrated by the author [5], for the case when the displacement components can be taken to vary linearly through the thickness. The significance of equations (1) lies in the fact that they allow the explicit integration, with respect to the dimensionless thickness coordinate, of the differential equations governing a first-approximation nonlinear theory corresponding to the scaling of stresses and displacements adopted in the next section for a thin plate.

For a transversely isotropic elastic body the nonlinear stress–displacement relations are

$$\left.\begin{aligned} s_{11}-\nu s_{22}-\nu_3 s_{33} &= E[v_{1,1}+\tfrac{1}{2}(v_{3,1})^2],\\ s_{22}-\nu s_{11}-\nu_3 s_{33} &= E[v_{2,2}+\tfrac{1}{2}(v_{3,2})^2],\\ 2(1+\nu)s_{12} &= E(v_{1,2}+v_{2,1}+v_{3,1}v_{3,2}),\\ s_{13} &= G(v_{1,3}+v_{3,1}),\\ s_{23} &= G(v_{2,3}+v_{3,2}),\\ \frac{E}{E_3}s_{33}-\nu_3(s_{11}+s_{22}) &= Ev_{3,3}. \end{aligned}\right\} \tag{4}$$

SCALING

When referring to an asymmetrically layered plate composed of two layers, hereafter called platings, of different thicknesses and materials, the original set of coordinates will be identified in turn with a set of right-handed Cartesian coordinates, x_i and x_i', such that $x_3 = 0$ and $x_3' = 0$ represent respectively the middle planes of the lower and upper platings. The latter then occupy the regions bounded by the two faces,

$$x_3 = -H/2, \qquad x_3 = H/2$$

and

$$x_3' = -H'/2, \qquad x_3' = H'/2,$$

respectively, and a cylindrical surface having generators normal to the middle planes.

Introduce, for the lower plating, the dimensionless coordinates

$$\xi = x_1/L, \qquad \eta = x_2/L, \qquad \zeta = 2x_3/H, \tag{5}$$

where L is a characteristic length along the circumference of the cylindrical surface such that the order of magnitude of H is small compared to L. Let

$$\lambda = H/2L \tag{6}$$

be the corresponding small expansion parameter.

For the thin lower plating, the stresses can then be scaled as

$$\left.\begin{aligned} &s_{11} = \sigma_0 s_x, \qquad s_{22} = \sigma_0 s_y, \qquad s_{12} = s_{21} = \sigma_0 t,\\ &s_{13} = s_{31} = \sigma_0\lambda t_x, \qquad s_{23} = s_{32} = \sigma_0\lambda t_y, \qquad s_{33} = \sigma_0\lambda^2 s, \end{aligned}\right\} \tag{7}$$

where σ_0 is a reference stress, and the displacements can be scaled as

$$Ev_1 = \sigma_0 uL, \qquad Ev_2 = \sigma_0 vL, \qquad Ev_3 = \sigma_0\lambda^{-1}wL. \tag{8}$$

Using equations (5)–(8), the three-dimensional nonlinear equilibrium equations become

$$\left.\begin{aligned} s_{x,\xi}+t_{,\eta}+t_{x,\zeta} &= 0,\\ t_{,\xi}+s_{y,\eta}+t_{y,\zeta} &= 0,\\ t_{x,\xi}+t_{y,\eta}+\frac{\sigma_0}{E\lambda^2}[(s_x w_{,\xi}+tw_{,\eta})_{,\xi}+(tw_{,\xi}+s_y w_{,\eta})_{,\eta}]+s_{,\zeta} &= 0. \end{aligned}\right\} \tag{9}$$

Assuming for simplicity the lower face of the lower plating to be free of stress, from equation (3) and equations (5)–(8), the face stress boundary conditions for the lower plating are

$$\left.\begin{aligned} \zeta = -1: &\quad t_x = t_y = s = 0, \\ \zeta = 1: &\quad t_x = T_x, \quad t_y = T_y, \quad s = S, \end{aligned}\right\} \tag{10}$$

where T_x, T_y, and S are respectively the dimensionless transverse shear and normal stresses acting on the lower plating at the bonding interface between the two layers.

Finally, using equations (5)–(8), the three-dimensional nonlinear stress–displacement relations lead to

$$\left.\begin{aligned} s_x - \nu s_y - \nu_3 \lambda^2 s &= u_{,\xi} + \frac{\sigma_0}{2E\lambda^2}(w_{,\xi})^2, \\ s_y - \nu s_x - \nu_3 \lambda^2 s &= v_{,\eta} + \frac{\sigma_0}{2E\lambda^2}(w_{,\eta})^2, \\ 2(1+\nu)t &= u_{,\eta} + v_{,\xi} + \frac{\sigma_0}{E\lambda^2} w_{,\xi} w_{,\eta}, \\ t_x &= \frac{G}{E\lambda^2}(u_{,\zeta} + w_{,\xi}), \\ t_y &= \frac{G}{E\lambda^2}(v_{,\zeta} + w_{,\eta}), \\ s - \frac{E_3 \nu_3}{E\lambda^2}(s_x + s_y) &= \frac{E_3}{E\lambda^4} w_{,\zeta}. \end{aligned}\right\} \tag{11}$$

Similarly, for the upper plating, let

$$\xi' = x_1'/L, \qquad \eta' = x_2'/L, \qquad \zeta' = 2x_3'/H' \tag{12}$$

be new dimensionless coordinates such that the order of magnitude of H' is small compared to L, and

$$\lambda' = H'/2L \tag{13}$$

be the corresponding small expansion parameter.

For the thin upper plating the stresses s_{ij}' can then be scaled as in equation (7), replacing all quantities by their primed counterparts, and similarly for the displacements v_i' in equation (8). Equations (9) and (11) hold for the primed quantities also. However, assuming for simplicity the upper face of the upper plating to be free of stress, the face stress boundary conditions for the upper plating now read

$$\left.\begin{aligned} \zeta' = 1: &\quad t_x' = t_y' = s' = 0, \\ \zeta' = -1: &\quad t_x' = T_x, \quad t_y' = T_y, \quad s' = S, \end{aligned}\right\} \tag{14}$$

when the conditions of the continuity of transverse shear and normal stresses at the bonding interface between the two layers are used.

PARAMETRIC EXPANSION AND FIRST-APPROXIMATION NONLINEAR THEORY OF LAYERED PLATE

All stresses, including the transverse shear and normal stress components at the bonding interface, and displacements are now expanded in powers of λ^2 for the lower plating.

$$\left.\begin{aligned} s_x &= \lambda^2 s_x^{(2)} + \lambda^4 s_x^{(4)} + \dots, \quad \dots, \quad T_x = \lambda^2 T_x^{(2)} + \lambda^4 T_x^{(4)} + \dots, \\ \dots, \quad S &= \lambda^2 S^{(2)} + \lambda^4 S^{(4)} + \dots, \quad \dots, \quad w = \lambda^2 w^{(2)} + \lambda^4 w^{(4)} + \dots, \end{aligned}\right\} \tag{15}$$

where $s_x^{(2)}$, $s_x^{(4)}$, etc., are regular functions of the independent variables except λ^2, since ξ, η, ζ have a range independent of λ^2 by definition. A similar set of relations can be written for the primed variables of the upper plating in terms of powers of λ'^2.

Substituting equation (15) into equations (9)–(11) and equating corresponding powers of λ^2 successive systems of differential equations can be obtained for the lower plating. From the coefficients of λ^2 a first-approximation nonlinear theory follows in the manner previously described [2], except that now the stress boundary conditions at the bonding interface must also be taken into account. The results can be summarized as follows.

$$\left.\begin{aligned} w^{(2)} &= w_0^{(2)}(\xi, \eta), \\ u^{(2)} &= u_0^{(2)}(\xi, \eta) - \zeta w_{0,\xi}^{(2)}, \\ v^{(2)} &= v_0^{(2)}(\xi, \eta) - \zeta w_{0,\eta}^{(2)}, \end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned} s_x^{(2)} &= (1-\nu^2)^{-1}\left\{u_{0,\xi}^{(2)} + \nu v_{0,\eta}^{(2)} + \frac{\sigma_0}{2E}[(w_{0,\xi}^{(2)})^2 + \nu(w_{0,\eta}^{(2)})^2] - \zeta(w_{0,\xi\xi}^{(2)} + \nu w_{0,\eta\eta}^{(2)})\right\}, \\ s_y^{(2)} &= (1-\nu^2)^{-1}\left\{v_{0,\eta}^{(2)} + \nu u_{0,\xi}^{(2)} + \frac{\sigma_0}{2E}[(w_{0,\eta}^{(2)})^2 + \nu(w_{0,\xi}^{(2)})^2] - \zeta(w_{0,\eta\eta}^{(2)} + \nu w_{0,\xi\xi}^{(2)})\right\}, \\ t^{(2)} &= \tfrac{1}{2}(1+\nu)^{-1}\left(u_{0,\eta}^{(2)} + v_{0,\xi}^{(2)} + \frac{\sigma_0}{E} w_{0,\xi}^{(2)} w_{0,\eta}^{(2)} - 2\zeta w_{0,\xi\eta}^{(2)}\right), \end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} t_x^{(2)} &= \tfrac{1}{2}(1+\zeta)T_x^{(2)} - \tfrac{1}{2}(1-\nu^2)^{-1}(1-\zeta^2)(\nabla^2 w_0^{(2)})_{,\xi}, \\ t_y^{(2)} &= \tfrac{1}{2}(1+\zeta)T_y^{(2)} - \tfrac{1}{2}(1-\nu^2)^{-1}(1-\zeta^2)(\nabla^2 w_0^{(2)})_{,\eta}, \\ s^{(2)} &= \tfrac{1}{2}(1+\zeta)\{S^{(2)} + \tfrac{1}{2}(1-\zeta)(T_{x,\xi}^{(2)} + T_{y,\eta}^{(2)})\} + \tfrac{1}{6}\zeta(1-\zeta^2)(1-\nu^2)^{-1}\nabla^2\nabla^2 w_0^{(2)} \\ &\quad - (1-\zeta^2)\frac{\sigma_0}{2E}(1-\nu^2)^{-1}\{(\nabla^2 w_0^{(2)})^2 + w_{0,\xi}^{(2)}(\nabla^2 w_0^{(2)})_{,\xi} \\ &\quad + w_{0,\eta}^{(2)}(\nabla^2 w_0^{(2)})_{,\eta} - 2(1-\nu)[w_{0,\xi\xi}^{(2)} w_{0,\eta\eta}^{(2)} - (w_{0,\xi\eta}^{(2)})^2]\}, \\ \nabla^2(\quad) &= (\quad)_{,\xi\xi} + (\quad)_{,\eta\eta}, \end{aligned}\right\} \tag{18}$$

$$\begin{aligned} u_{0,\xi\xi}^{(2)} + \tfrac{1}{2}(1-\nu)u_{0,\eta\eta}^{(2)} + \tfrac{1}{2}(1+\nu)v_{0,\xi\eta}^{(2)} &= -\tfrac{1}{2}(1-\nu^2)T_x^{(2)} - \frac{\sigma_0}{2E}\{(1-\nu)w_{0,\xi}^{(2)}\nabla^2 w_0^{(2)} \\ &\quad + \tfrac{1}{2}(1+\nu)[(w_{0,\xi}^{(2)})^2 + (w_{0,\eta}^{(2)})^2]_{,\xi}\}, \end{aligned}$$

$$
\left.\begin{aligned}
& v_{0,\eta\eta}^{(2)}+\tfrac{1}{2}(1-\nu)v_{0,\xi\xi}^{(2)}+\tfrac{1}{2}(1+\nu)u_{0,\xi\eta}^{(2)} \\
&= -\tfrac{1}{2}(1-\nu^2)T_y^{(2)}-\frac{\sigma_0}{2E}\{(1-\nu)w_{0,\eta}^{(2)}\nabla^2 w_0^{(2)} \\
&\quad +\tfrac{1}{2}(1+\nu)[(w_{0,\xi}^{(2)})^2+(w_{0,\eta}^{(2)})^2]_{,\eta}\}, \\
\nabla^2\nabla^2 w_0^{(2)} &= \tfrac{3}{2}(1-\nu^2)(T_{x,\xi}^{(2)}+T_{y,\eta}^{(2)}+S^{(2)}) \\
&\quad +\frac{3\sigma_0}{E}\left\langle w_{0,\xi\xi}^{(2)}\left\{u_{0,\xi}^{(2)}+\nu v_{0,\eta}^{(2)}+\frac{\sigma_0}{2E}[(w_{0,\xi}^{(2)})^2\right.\right. \\
&\quad \left.+\nu(w_{0,\eta}^{(2)})^2]\right\}+w_{0,\eta\eta}^{(2)}\left\{v_{0,\eta}^{(2)}+\nu u_{0,\xi}^{(2)}\right. \\
&\quad \left.+\frac{\sigma_0}{2E}[(w_{0,\eta}^{(2)})^2+\nu(w_{0,\xi}^{(2)})^2]\right\}+w_{0,\xi\eta}^{(2)}(1-\nu) \\
&\quad \left.\cdot\left(u_{0,\eta}^{(2)}+v_{0,\xi}^{(2)}+\frac{\sigma_0}{E}w_{0,\xi}^{(2)}w_{0,\eta}^{(2)}\right)\right\rangle.
\end{aligned}\right\} \quad (19)
$$

The equilibrium equations of the first-approximation nonlinear theory can also be expressed in terms of the corresponding dimensionless stress and couple resultants. For the lower plating, these are

$$
\left.\begin{aligned}
N_{x,\xi}^{(2)}+N_{,\eta}^{(2)}+T_x^{(2)} &= 0, \\
N_{,\xi}^{(2)}+N_{y,\eta}^{(2)}+T_y^{(2)} &= 0, \\
Q_{x,\xi}^{(2)}+Q_{y,\eta}^{(2)}+\frac{\sigma_0}{E}[(N_x^{(2)}w_{0,\xi}^{(2)}+N^{(2)}w_{0,\eta}^{(2)})_{,\xi}+(N^{(2)}w_{0,\xi}^{(2)}+N_y^{(2)}w_{0,\eta}^{(2)})_{,\eta}]+S^{(2)} &= 0, \\
M_{x,\xi}^{(2)}+M_{,\eta}^{(2)}-Q_x^{(2)}+T_x^{(2)} &= 0, \\
M_{,\xi}^{(2)}+M_{y,\eta}^{(2)}-Q_y^{(2)}+T_y^{(2)} &= 0,
\end{aligned}\right\} \quad (20)
$$

where

$$
\left.\begin{aligned}
N_x^{(2)} &= \int_{-1}^{1} s_x^{(2)}\,d\zeta = 2(1-\nu^2)^{-1}\left\{u_{0,\xi}^{(2)}+\nu v_{0,\eta}^{(2)}+\frac{\sigma_0}{2E}[(w_{0,\xi}^{(2)})^2+\nu(w_{0,\eta}^{(2)})^2]\right\}, \\
N_y^{(2)} &= \int_{-1}^{1} s_y^{(2)}\,d\zeta = 2(1-\nu^2)^{-1}\left\{v_{0,\eta}^{(2)}+\nu u_{0,\xi}^{(2)}+\frac{\sigma_0}{2E}[(w_{0,\eta}^{(2)})^2+\nu(w_{0,\xi}^{(2)})^2]\right\}, \\
N^{(2)} &= \int_{-1}^{1} t^{(2)}\,d\zeta = (1+\nu)^{-1}\left(u_{0,\eta}^{(2)}+v_{0,\xi}^{(2)}+\frac{\sigma_0}{E}w_{0,\xi}^{(2)}w_{0,\eta}^{(2)}\right), \\
Q_x^{(2)} &= \int_{-1}^{1} t_x^{(2)}\,d\zeta = T_x^{(2)}-\tfrac{2}{3}(1-\nu^2)^{-1}(\nabla^2 w_0^{(2)})_{,\xi}, \\
Q_y^{(2)} &= \int_{-1}^{1} t_y^{(2)}\,d\zeta = T_y^{(2)}-\tfrac{2}{3}(1-\nu^2)^{-1}(\nabla^2 w_0^{(2)})_{,\xi}, \\
M_x^{(2)} &= \int_{-1}^{1} s_x^{(2)}\zeta\,d\zeta = -\tfrac{2}{3}(1-\nu^2)^{-1}(w_{0,\xi\xi}^{(2)}+\nu w_{0,\eta\eta}^{(2)}),
\end{aligned}\right\} \quad (21)
$$

$$\left.\begin{aligned} M_y^{(2)} &= \int_{-1}^{1} s_y^{(2)}\zeta\,\mathrm{d}\zeta = -\tfrac{2}{3}(1-\nu^2)^{-1}(w_{0,\eta\eta}^{(2)}+\nu w_{0,\xi\xi}^{(2)}),\\ M^{(2)} &= -\tfrac{2}{3}(1+\nu)^{-1}w_{0,\xi\eta}^{(2)}. \end{aligned}\right\}$$

Relations similar to equations (16)–(19) can be obtained for the upper plating from the coefficients of λ'^2 in the corresponding parametric expansion or, simply, by replacing all quantities in equations (16)–(19) by their primed counterparts, and T_x, T_y, and S in equations (18) and (19) by $-T_x$, $-T_y$, and $-S$, respectively. Analogous remarks hold for equations (20) and (21).

From the conditions of continuity of displacements at the bonding interface, and using the transformation

$$\xi' = \xi, \qquad \eta' = \eta \tag{22}$$

in equation (16) and its counterpart for the upper plating, it follows that

$$\left.\begin{aligned} &w_0'^{(2)} = w_0^{(2)},\\ &u_0'^{(2)} = u_0^{(2)}-2w_{0,\xi}^{(2)}, \qquad v_0'^{(2)} = v_0^{(2)}-2w_{0,\eta}^{(2)}. \end{aligned}\right\} \tag{23}$$

Substituting equations (22) and (23) into the counterpart of equation (19) for the upper plating, and eliminating T_x, T_y, and S between the resulting set of equations and equation (19), the following system of three differential equations in terms of the three unknowns $u_0^{(2)}$, $v_0^{(2)}$, and $w_0^{(2)}$ are obtained.

$$\left.\begin{aligned} &u_{0,\xi\xi}^{(2)}+\frac{1}{1+\alpha}[\tfrac{1}{2}(1+\beta)(1-\nu)u_{0,\eta\eta}^{(2)}+\tfrac{1}{2}(1+\alpha/\beta)(1+\nu)v_{0,\xi\eta}^{(2)}]\\ &\quad= -\frac{\sigma_0}{2E(1+\alpha)}\{(1+\beta\delta)(1-\nu)w_{0,\xi}^{(2)}\nabla^2 w_0^{(2)}+\tfrac{1}{2}(1+\alpha\delta/\beta)(1+\nu)\\ &\quad\cdot[(w_{0,\xi}^{(2)})^2+(w_{0,\eta}^{(2)})^2]_{,\xi}\}+\frac{2\alpha}{1+\alpha}(\nabla^2 w_0^{(2)})_{,\xi},\\ &v_{0,\eta\eta}^{(2)}+\frac{1}{1+\alpha}[\tfrac{1}{2}(1+\beta)(1-\nu)v_{0,\xi\xi}^{(2)}+\tfrac{1}{2}(1+\alpha/\beta)(1+\nu)u_{0,\xi\eta}^{(2)}]\\ &\quad= -\frac{\sigma_0}{2E(1+\alpha)}\{(1+\beta\delta)(1-\nu)w_{0,\eta}^{(2)}\nabla^2 w_0^{(2)}+\tfrac{1}{2}(1+\alpha\delta/\beta)(1+\nu)\\ &\quad\cdot[(w_{0,\xi}^{(2)})^2+(w_{0,\eta}^{(2)})^2]_{,\eta}\}+\frac{2\alpha}{1+\alpha}(\nabla^2 w_0^{(2)})_{,\eta},\\ &\nabla^2\nabla^2 w_0^{(2)} = \frac{3\sigma_0}{E(1+\alpha)}\Big\langle(1+\alpha\delta)(w_{0,\xi\xi}^{(2)}u_{0,\xi}^{(2)}+w_{0,\eta\eta}^{(2)}v_{0,\eta}^{(2)})+(1+\alpha\delta\varepsilon)\nu(w_{0,\xi\xi}^{(2)}v_{0,\eta}^{(2)}+w_{0,\eta\eta}^{(2)}u_{0,\xi}^{(2)})\\ &\quad+\frac{\sigma_0}{2E}\{(1+\alpha\delta^2)[w_{0,\xi\xi}^{(2)}(w_{0,\xi}^{(2)})^2+w_{0,\eta\eta}^{(2)}(w_{0,\eta}^{(2)})^2+2w_{0,\xi\eta}^{(2)}w_{0,\xi}^{(2)}w_{0,\eta}^{(2)}]\\ &\quad+(1+\alpha\delta^2\varepsilon)\nu[w_{0,\xi\xi}^{(2)}(w_{0,\eta}^{(2)})^2+w_{0,\eta\eta}^{(2)}(w_{0,\xi}^{(2)})^2]\}\\ &\quad+(1+\beta\delta)(1-\nu)w_{0,\xi\eta}^{(2)}(u_{0,\eta}^{(2)}+v_{0,\xi}^{(2)})-4\alpha\delta[(w_{0,\xi\xi}^{(2)})^2\\ &\quad+(w_{0,\eta\eta}^{(2)})^2+\varepsilon\nu w_{0,\xi\xi}^{(2)}w_{0,\eta\eta}^{(2)}+\frac{\beta}{\alpha}(1-\nu)(w_{0,\xi\eta}^{(2)})^2]\Big\rangle, \end{aligned}\right\} \tag{24}$$

$$\alpha = \frac{1-\nu^2}{1-\nu'^2}, \qquad \beta = \frac{1+\nu}{1+\nu'}, \qquad \delta = \frac{\sigma_0' E}{\sigma_0 E'}, \qquad \varepsilon = \frac{\nu'}{\nu}.$$

Equations (24) constitute the differential equations of a von Kármán-type first-approximation nonlinear theory for a thin composite elastic plate, made of two dissimilar and transversely isotropic layers, in terms of the displacement components defined at the middle plane of the lower plating. The corresponding equations for the classical case of a homogeneous thin elastic plate undergoing moderately large deflection are given, for instance, by Chien and Yeh [6], in terms of the displacement components defined at the middle plane of the plate.

A special case of interest is when $\varepsilon = 1$, so that $\alpha = 1$ and $\beta = 1$ also, with $\delta \neq 1$. The various coefficients in equations (24) then simplify considerably for a layered plate made of two platings having nearly equal Poisson's ratios but being otherwise dissimilar.

CLOSURE

A first-approximation theory governing the moderately large deflection of an asymmetrically layered transversely isotropic plate has been obtained from a three-dimensional partially nonlinear theory of elasticity using the Lagrangian description and a parametric expansion of the interior solution. A higher-order approximation can be obtained from the next system of equations corresponding to the coefficients of λ^4 and λ'^4 in that expansion. These equations would then introduce the effects of transverse shear and normal stresses in the form of terms containing E_3, ν_3 and E_3', ν_3'. As already noted elsewhere [2] for the case of a homogeneous plate, the limiting case of small deflection of a layered plate can also be obtained from the first-approximation theory given here by formally setting the ratio σ_0/E equal to zero. In fact, a further study [7] of elastic plate theory has revealed that σ_0/E, when taken as a small parameter, σ, may be used to extract various successive approximations, depending on the relative magnitudes of the two parameters λ^2 and σ, from the general nonlinear theory of elasticity in the reference state.

The present investigation can be complemented by suitable edge boundary conditions as done, for instance, by Reissner [8] for the linear theory of homogeneous plates. The complete set of successive systems of two-dimensional equations would then be solvable by standard methods.

An idea of some of the limitations of our formulation can be gained from a recent evaluation of similar methods of approximation extensively used in the study of liquid surface waves [9].

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Résumé—Une technique paramétrique de développement est employée pour extraire des systèmes d'équations à deux dimensions régissant la déformation légèrement accentuée d'une plaque élastique transversalement isotropique et à couches asymétriques, d'une théorie d'élasticité à trois dimensions, en partie non-linéaire en fonction d'un état de référence ou de la description de Lagrange. Les équations de cette dernière théorie sont d'abord exprimées sous forme sans dimension pour chaque couche mince au moyen d'une graduation adéquate des efforts et déplacements contenant le paramètre de développement. Des systèmes successifs d'équations différentielles sont alors obtenus en mettant sous forme d'équations des puissances correspondantes de ce dernier après avoir développé toutes les variables paramétriquement. Le premier de ces systèmes produit une théorie de première approximation, type von Kármán, non-linéaire, d'une plaque en couches lorsque toutes les conditions de continuité au contact des faces internes de liaison entre les diverses couches sont prises en considération. Pour une plaque faite de deux couches non similaires sont présentées les équations différentielles en termes des déplacements définis au plan médian d'une des couches. Des approximations systématiques de plus grand ordre peuvent être obtenues des systèmes suivants d'équations. Le premier de tels systèmes montrerait les effets d'un cisaillement transversal et les fatigues normales sur la déformation.

Zusammenfassung—Eine parametrische Entwicklungstechnik wird angewendet um zweidimensionale Gleichungen für die Durchbiegung isotropischer Platten von der teilweise nichtlinearen dreidimensionalen Elastizitätstheorie zu erhalten die als Lagrange Beschreibung dargestellt werden kann. Die Gleichungen dieser Theorie werden vorerst für jede dünne Schicht dimensionslos ausgedrückt indem die Spannungen und Bewegungen reduziert werden. Stufenweise werden dann Differenzialgleichungen erhalten indem man entspechende Kräfte gleichsetzt nachdem alle Variabeln parametrisch entwickelt wurden. Das erste System dieser Art gibt eine Annäherung einer Kármán artigen Lineartheorie wenn alle Grenzbedingungen berücksichtigt werden. Im Falle einer Platte aus zwei verschiedenen Schichten wird die Differenzialgleichung für die Mittelebene einer Schicht gegeben. Systematische Annäherungswerte höherer Ordnung können aus der nächsten Gleichungssystemen erhalten werden. Das erste dieser Systeme zeigt die Resultate normaler Spannung und transversaler Scherung auf die Durchbiegung.

Абстракт—Применяется техника параметрического расширения для систем извлечения двумерных уравнений, управляющих умеренно большим отклонением эластичной ассиметрически прослоенной поперечно изотропной пластины из тримерной, частично нелинейной теории эластичности в терминах ссылочного положения или Лагранжевой дескрипции. Уравнения последней теории сначала выражены в безразмерной форме для каждого тонкого слоя посредством подходящего пересчёта напряжений и смещений, включающих параметр расширения. Последовательные системы дифференциальных уравнений получаются затем составлением уравнений соответствующих сил последних, после того, как все переменные расширены параметрически. Первая такая система поддаётся первому приближению типа фон Кàрман (von Kármán) теории нелинейно прослоенной пластины, когда приняты во внимание все условия непрерывности у связи поверхности раздела между слоями. Для пластины, сделанной из двух неоднородных слоёв представлены соответствующие дифференциальные уравнения в терминах перемещений, определённых в средней плоскости одного из слоёв. Систематические приближения высшего порядка могут быть получены из последующих систем уравнений. Первая такая система выявила бы эффекты поперечного сдвига и нормальных напряжений на деформацию.